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Analytical Gain Scheduling Approach to Periodic Observer Design

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A method for analytic synthesis of periodic observer gains for linear periodic systems is presented. The eigenvalues of the transition matrix after one period (the Floquet poles) may be assigned; thus the method is similar to synthesis of time-invariant observers by pole placement. The closed-loop transition matrix is also obtained analytically. The method is demonstrated on a linearized model of a satellite in an eccentric orbit using a rate gyro and a horizon sensor to estimate pitch attitude and orbit eccentricity e. Since the gains are expressed as analytic functions of e, they can be changed as e drifts due to orbit perturbations. This involves far less computation and storage than an extended Kalman filter.

Introduction

EARTH satellite orbits are slowly changing periodic (SCP), i.e., they are approximately periodic but change slowly due to the non-Keplerian perturbing forces of Earth, the atmosphere, the sun, and the moon. The orbital elements can be estimated from observations using an extended Kalman filter (EKF), but this requires considerable computation and storage since the covariance matrix must be propagated in real time in order to obtain the filter gains; these gains are also SCP.

Orbital elements cannot be estimated using observers with constant gains. A method for synthesizing periodic observer gains with specified Floquet poles is given in this paper. Implementation of such an observer greatly reduces the computation load compared to the use of an EKF with only a small loss in estimate accuracy. This is of special interest for autonomous navigation (autonav) schemes being considered for many future spacecraft.

Simple Example

One frequently studied autonav configuration uses an Earth sensor, rate gyros, and star trackers in a single satellite. A simplified version uses only an Earth sensor and a rate gyro to estimate orbit eccentricity e and satellite pitch attitude θ (see Fig. 1). For simplicity, the other orbital elements are assumed known by other means. The satellite is kept nearly nadir pointing by a closed-loop control system. The Earth sensor makes noisy measurements of the pitch angle θ and the gyro makes noisy measurements of inertial pitch rate q.

The estimator consists of a kinematic model of the system linearized about the current estimate of eccentricity e. Appendix A shows that the estimate errors $\tilde{e}$, $\tilde{\theta}$ are described by

$$\begin{bmatrix} \tilde{\theta}' \\ \tilde{e}' \end{bmatrix} = \begin{bmatrix} -K_\theta(\nu) & B(e)C(e,\nu) \\ -K_e(\nu) & 0 \end{bmatrix} \begin{bmatrix} \tilde{\theta} \\ \tilde{e} \end{bmatrix} + \begin{bmatrix} -w + K_\theta v \\ K_e v \end{bmatrix} \tag{1}$$

$$B = \frac{2+e}{1-e^2} \tag{2}$$

$$C = \frac{2\cos(\nu) + e\cos(2\nu)}{2+e} \tag{3}$$

where the independent variable ν is true anomaly, the prime indicates derivative with respect to ν, and (w, v) are errors in the (rate gyro, Earth sensor) measurements. We wish to synthesize observer gains $K_\theta(v)$, $K_e(v)$ to stabilize this system, i.e., to attenuate the estimate errors.

This model is much simpler than the model for the full autonav system, but it retains its key features:

1) For several orbits, e changes only slightly and $C(e, \nu)$ is a periodic function of ν.

2) Over many periods, e drifts due to perturbation forces and orbit-adjust maneuvers. The variation of C with ν changes significantly with changes in e (see Fig. 2).

3) Even treating e as constant, constant gains cannot stabilize the system.

Recursive least-squares (RLS) methods are commonly used for problems like this one. However, RLS methods require excessive computation and storage; we seek methods that reduce these requirements. First, we show that, as asserted, no set of constant gains can stabilize the system. Then we present a new method of analytical gain scheduling (AGS). Finally, we compare AGS with RLS.

Failure of Constant Gains

In Eq. (1), assume K_θ and K_e are constants. We consider the transition matrix $\Phi(\nu)$ and show that either its (2, 1) element $\Phi_{e\theta}$ or its (2, 2) element Φ_{ee} does not decay to zero with increasing ν; hence, the closed-loop system is unstable.

Suppose that Φ has the form

$$\Phi(\nu) = \begin{bmatrix} e^{f_4(\nu)} & f_3(\nu) \\ e^{f_4(\nu)} f_1(\nu) & e^{f_2(\nu)} + f_1(\nu) f_3(\nu) \end{bmatrix} \tag{4}$$

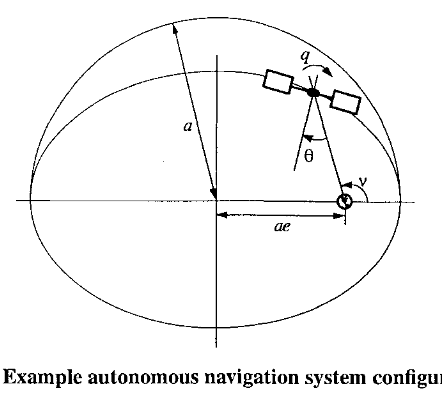

Fig. 1 Example autonomous navigation system configuration.

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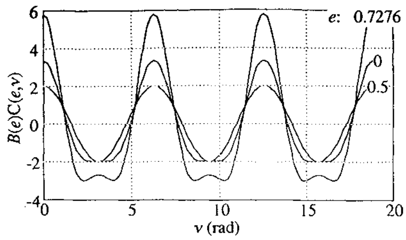

Fig. 2 Coefficient $B(e)C(e, \nu)$ for various values of e.

for some functions $f_i(\nu), i = 1, \ldots, 4$. (The motivation for this form is given in the next section.) This expression may be regarded as the result of a nonlinear coordinate transformation from the four-dimensional space of elements of Φ to the four-dimensional space of f_i. Since Φ must satisfy[1]

$$\Phi' = \begin{bmatrix} -K_\theta(\nu) & BC(\nu) \\ -K_e(\nu) & 0 \end{bmatrix} \Phi \tag{5}$$

$$\Phi(0) = I \tag{6}$$

direct substitution shows that f_i must satisfy

$$f_1'(\nu) = -K_e + K_\theta f_1(\nu) - BCf_1^2(\nu) \tag{7}$$

$$f_2'(\nu) = -BCf_1(\nu) \tag{8}$$

$$f_3'(\nu) = BC\big(e^{f_2(\nu)} + f_1(\nu)f_3(\nu)\big) - K_\theta f_3(\nu) \tag{9}$$

$$f_4'(\nu) = -K_\theta + BCf_1(\nu) \tag{10}$$

$$f_i(0) = 0, \qquad i = 1, \ldots, 4 \tag{11}$$

If $K_e = 0$, Eqs. (11), (7), and (8) show that $f_1(\nu) = f_2(\nu) = 0$ and, according to Eqn. (4), Φ_{ee} does not decay with ν. If $K_e \neq 0$, Eqs. (4) and (5) show that

$$\Phi_{e\theta}' = -K_e e^{f_4} \tag{12}$$

is either positive or negative for all ν. Since, from Eq. (6) [or Eq. (4)], $\Phi_{e\theta}(0) = 0$, $\Phi_{e\theta}$ is either positive or negative for all $\nu > 0$.

Wei-Norman Theory and Fdot Equations

A systematic approach, generalizable to higher order systems, was used to find the transformation of Eq. (4). It is based on the result of Wei and Norman[2] for the solution of the general, linear, time-varying ordinary differential equation.

Given

$$x(t) = \Phi(t)x(0) \tag{13}$$

$$\dot{\Phi} = F(t)\Phi \tag{14}$$

$$\Phi(0) = I \tag{15}$$

Wei and Norman considered solutions of the form

$$\Phi = \prod_{i=1}^{m} \Phi_i(t) \tag{16}$$

$$\Phi_i(t) = \exp[F_i f_i(t)] \tag{17}$$

for some integer m, constant matrices F_i, and scalar functions of time $f_i, i = 1, m$. From differentiation and substitution into Eq. (14), Eqs. (16) and (17) are seen to hold if

$$\sum_{j=i}^{m} \left[\left(\prod_{i=1}^{j-1} \Phi_i \right) \dot{\Phi}_j \left(\prod_{i=j+1}^{m} \Phi_i \right) \right] = F\left(\prod_{i=1}^{m} \Phi_i \right) \tag{18}$$

or, more concisely,

$$\sum_{j=1}^{m} \dot{f}_j \Psi_{j-1} F_j \Psi_{j-1}^{-1} = F \tag{19}$$

where

$$\Psi_j(t) = \prod_{i=1}^{j} \Phi_i(t) \qquad j \geq 1 \qquad \Psi_0(t) = I \tag{20}$$

This set of n^2 equations, linear in the $m \dot{f}_i$, will be called the Fdot equations.

Wei and Norman proved that there always exists a set of $m F_i$ such that their associated Fdot equations may be inverted, yielding an mth-order system of first-order ordinary differential equations that are in general nonlinear, coupled, and time varying. The f_i may be thought of as an m-dimensional coordinate set related to x by the nonlinear transformation of Eqs. (13), (16), and (17). One set of initial conditions that always satisfies Eq. (15) is

$$f_i(0) = 0, \qquad i = 1, \ldots, m \tag{21}$$

However, there may be other sets.

Example. Eq. (4) was found by trying all combinations of $m \leq 4$ and $F_i = O_{jk}$, where O_{jk} is a 2×2 matrix with its (j, k) element equal to 1 and its other elements equal to zero. The combination chosen is

$$m = 4 \tag{22}$$

$$F_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{23}$$

$$F_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{24}$$

$$F_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{25}$$

$$F_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{26}$$

The Fdot equations for the system are Eqs. (7–10). For this set of F_i, only the initial conditions shown in Eq. (21) satisfy Eq. (15).

A way to choose the smallest workable m or the F_i is not known by the authors. Wei and Norman proved that a given set of F_i yields invertable Fdot equations if, for every pair F_i and F_j in the set, the commutator product $F_iF_j - F_jF_i$ is also in the set; therefore, from linear algebra, there is always a workable m no larger than n_s^2, where n_s is the system order. For sparse F, a much smaller m may do. With perseverance, one can usually find a set of F_i that yields Fdot equations that, after inversion, are one-way coupled. Searching for such a set manually would be too tedious to be practical, but the process can be automated with symbol manipulation software.

Unfortunately, even if one-way coupled, the inverted Fdot equations are in general nonlinear and time varying and typically no easier to solve than the original coupled equations. Thus, for the example system with constant gains, the Fdot equations allow one to infer that the system is unstable, but do not allow to solve for the motion. This is probably the reason the Wei-Norman approach has received little attention in the literature.

Analytical Gain Scheduling Approach

Feedback synthesis gives the analyst an additional freedom that renders Wei-Norman theory useful. Even if one cannot solve for the open-loop response of the given system, it may be possible to constrain the closed-loop response and determine the feedback gains required to obtain that response.

Example. Consider Eq. (4) with the gains no longer constrained to be constant. If the closed-loop system is to be stable, f_2 and f_4 must tend to minus infinity and f_3 must decay to zero as υ approaches infinity; however f_1 may take on any finite value. Moreover, f_1 and one other of the f_i may be specified by the designer. Then the other two f_i and the gains are determined by Eqs. (7–10).

Some insight is required to choose f_1; many different choices will yield desirable closed-loop behavior, but most do not allow solving for the other f_i in closed form. We choose

$$f_1 = \frac{2(1-e^2)(2+e)}{4-2e+e^2} b[1-\cos(\nu)]C \tag{27}$$

where b is an inverse time constant to be selected later. The first quotient is inserted to conveniently scale the final result. The sinusoidal term is inserted to satisfy the initial condition. The C term is such that substitution into Eq. (8) yields

$$f_2' = -\frac{2(2+e)^2}{4-2e+e^2} b[1-\cos(\nu)]C^2 \le 0 \tag{28}$$

and f_2 tends to minus infinity as required.

Either f_3 or f_4 may still be chosen, and again some insight is needed; many different choices will yield desirable closed-loop behavior, but most require a substantial amount of algebra. One obvious simplification is

$$f_4 = f_2 \tag{29}$$

which allows Eq. (9) to be written as

$$f_3' = f_2' f_3 + BCe^{f_2} \tag{30}$$

so that

$$f_3 = De^{f_2} \tag{31}$$

where

$$D(e,\nu) = \int_0^\nu BC\,d\nu = \frac{4\sin(\nu) + e\sin(2\nu)}{2(1-e^2)} \tag{32}$$

and f_3 decays to zero as required. Equations (10), (29), (28), (27), and (7) together determine the gains as

$$K_\theta = \frac{4(2+e)^2}{4-2e+e^2} b[1-\cos(\nu)]C^2 \tag{33}$$

$$K_e = \frac{(1-e^2)(2+e)}{4-2e+e^2} b([1-\cos(\nu)](CK_\theta - 2C') - 2C\sin(\nu)) \tag{34}$$

where

$$C' = -2\frac{\sin(\nu) + e\sin(2\nu)}{2+e} \tag{35}$$

Equations (33) and (34) are plotted for various values of e in Fig. 3 and various values of b in Fig. 4. The gains tend to increase with b; however, the figures show that changing either b or e changes not only the amplitude but also the frequency content of the gains. It is unlikely that an analyst would guess a similar structure.

The closed-loop transition matrix may now be expressed as a function of ν and e. First Eq. (28) is integrated to yield

$$f_2 = -b\left(\nu + \sum_{n=1}^{5} a_n \sin(n\nu)\right) \tag{36}$$

where

$$a_1 = -\frac{6-4e+e^2}{4-2e+e^2} \tag{37}$$

$$a_2 = \frac{2(1-e)}{4-2e+e^2} \tag{38}$$

$$a_3 = -\frac{4-8e+e^2}{6(4-2e+e^2)} \tag{39}$$

$$a_4 = -\frac{e(2-e)}{4(4-2e+e^2)} \tag{40}$$

$$a_5 = -\frac{e^2}{10(4-2e+e^2)} \tag{41}$$

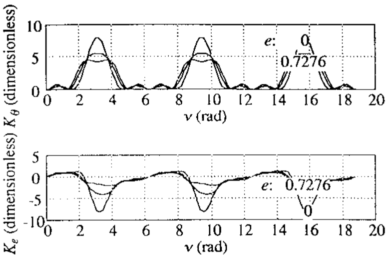

Fig. 3 AGS filter gains for various values of e ($b = 1$).

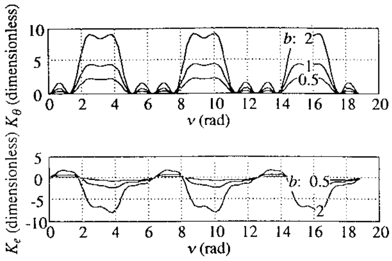

Fig. 4 AGS filter gains for various values of b ($e = 0.7276$).

Then substitutions are made into Eq. (4) from Eqs. (27), (36), (31), (32), and (29).

Another way to express Φ is

$$\Phi = \begin{bmatrix} 1 & D \\ f_1 & 1 + f_1 D \end{bmatrix} \begin{bmatrix} e^{f_2} & 0 \\ 0 & e^{f_2} \end{bmatrix} \tag{42}$$

This is a Floquet-like decomposition (a periodic component and a decaying component). The Floquet poles p_F are defined as

$$p_F = \text{eig}\begin{bmatrix} e^{f_2} & 0 \\ 0 & e^{f_2} \end{bmatrix}_{\nu=2\pi} = e^{-2\pi b},\ e^{-2\pi b} \tag{43}$$

Thus, b plays the same role that a pole plays in a time-invariant system; it may be chosen by the designer to satisfy performance and stability criteria. [There are really two "time-varying poles"; they were selected to be equal in Eq. (29).]

Because it results in analytical expressions for K_θ, K_e, and Φ, we call this procedure analytical gain scheduling (AGS). In principle, a similar process may be applied to any time-varying linear feedback system. The introduction of f_i coordinates 1) provides the slack to arbitrarily constrain the response of some coordinates and 2) makes it clear what sort of constraints are useful. The algorithm is presented in Fig. 5. The biggest drawback of AGS is that it requires searching for a suitable set of m and F_i, $i = 1, \ldots, m$. Some tools that are useful for the search are given in Appendix B.

AGS reflects a contrast between time-varying and time-invariant linear systems: In the time-invariant case, a general solution to Eq. (14) is known and therefore the same theory is useful for both solution of open-loop equations and synthesis of feedback gains. In the time varying case, the Wei-Norman theory yields a method for synthesizing feedback gains but is not useful for solving open-loop equations of motion.

Comparison with Recursive Least-Squares

Time-varying linear estimation problems are typically solved by RLS methods, that is, by the EKF or some approximation to it. In the EKF, gains are computed from a covariance estimate P that is propagated along with the state estimate. For linear periodic systems, EKF gains typically converge to a periodic steady state that can be computed a priori and fit to simple curves. This approximation technique will be called numerical gain scheduling (NGS).

In the EKF[3], P is propagated by

$$\dot{P} = FP + PF^T + Q - PH^TR^{-1}HP \tag{44}$$

Table 1 Input parameters for simulation of EKF

Symbol	Definition	Value
e	Eccentricity	0.7276
P_0	Initial estimate error covariance	diag$[(7\times10^{-4})^2\ (7\times10^{-4})^2]$
Q	Plant noise weighting matrix	diag$[(3.3\times10^{-4})^2\ (3.3\times10^{-4})^2]$/rev
R	Measurement noise weighting matrix	$(0.01)^2$
T	Orbit period	75,758 s
T_s	Sample period	0.00132 rev (100 s)
$\bar{x}_0$	Initial-state estimate error	$[5\times10^{-4}\ 5\times10^{-4}]^T$

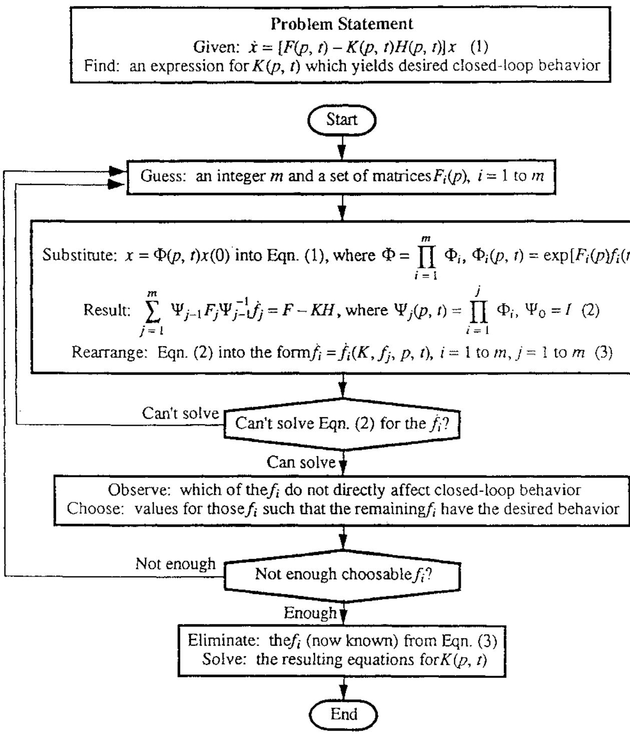

Fig. 5 AGS gain synthesis algorithm.

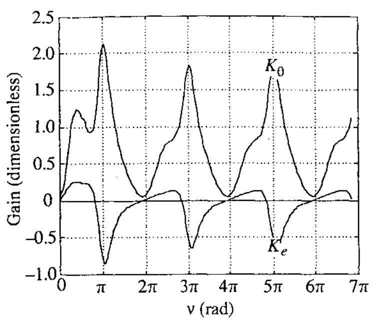

Fig. 6 EKF gains for the example ($e = 0.7276$).

and the gain is computed from

$$K^{\mathrm{EKF}} = PH^TR^{-1} \tag{45}$$

where F and H are the state and output matrices, respectively, and Q and R are weighting matrices chosen by the designer. In general, the EKF is finely tuned to F, H, Q, and R and is sensitive in both performance and stability to differences between the model and the physical system.

For onboard navigation, EKF performance is typically not sensitive to modeling errors because 1) there are a relatively large number of measurements, 2) the states are relatively well behaved, and 3) the estimates are not used for prediction. To save computation, disturbances due to the sun, the moon, atmospheric drag, etc., are neglected in the model. The model inaccuracy is accounted for by increasing Q. A nonzero Q is essential to reach a periodic, nonzero steady state.

In contrast, EKF stability is sensitive to modeling. Significant differences in the gains are observed, depending on whether Q is taken as constant in the time domain or the ν domain and on whether R increases with decreasing altitude, as it should for an Earth sensor. The ability to predict stability analytically is an advantage AGS holds over all RLS methods.

Figure 6 shows the gains computed when Eqs. (44) and (45) are applied to the example system. The matrices F, H, and Q are

$$F = \begin{bmatrix} 0 & BC \\ 0 & 0 \end{bmatrix} \tag{46}$$

$$H = [1 \quad 0] \tag{47}$$

$$Q = \begin{bmatrix} Q_\theta & 0 \\ 0 & Q_e \end{bmatrix} \tag{48}$$

The curves are obtained by transforming the results of a nonlinear, discrete, time-domain simulation. Both Q and R are taken as constant in the time domain. The values of Q/R are chosen so that the gains roughly match those of Fig. 3 in amplitude. Parameter values are shown in Table 1.

Within one orbit the gains reach a periodic steady state. For $e = 0$ (not shown), the EKF gains are strikingly similar to the AGS gains, the only difference being a twice-per-orbit component in K_θ^{AGS}. For high e, more differences develop, but the EKF gains still tend to support the frequency content of the AGS gains. In particular, both filters in both cases show little or no DC component for K_e, supporting the argument that K_e must change sign to stabilize the system.

The NGS estimator is designed by fitting curves to Fig. 6. Clearly, fitting enough terms of a Fourier series brings the steady-state performance of the NGS estimator arbitrarily close to that of the EKF. From inspection of the figure, only three terms are required for near-circular orbits, but for high-e orbits, the proper number of terms may only be determined by trial and error. This is the biggest drawback of the NGS approach. Also, for the full-up autonav problem, the system is periodic over the short term but drifts over the long term. A complete NGS estimator design is needed for each of several short-term nominal orbits, and the separate estimators must be connected by some type of interpolation. The design effort involved quickly becomes unreasonable. The overhead for such a system may also start to be significant.

The relative merits of AGS, EKF, and NGS are summarized in Table 2. Gain scheduling, numerical or analytical, can approach optimal steady-state estimate errors with low computational overhead and is therefore preferred. AGS has the advantage over NGS of admitting closed-form stability and performance evaluation. However, AGS solutions are difficult to find. For low-e systems, NGS has the advantage of much less design effort. For high-e systems, NGS requires carrying an excessive number of terms. For systems that are to

Table 2 Comparison of gain synthesis methods

Method	EKF	NGS	AGS
Design effort	Minimal	Moderate	Not always solvable, insight required
Analysis available	Time domain only	Time domain only	Analytical expression for transition matrix allows analog of frequency-domain analysis
Estimate error	Optimal	Arbitrarily close to optimal in steady state	Arbitrarily close to optimal in steady state
Computational overhead	Adds and multiplies grow as n_x^3; storage grows as n_x^2	Depends on functions chosen; typically small	Depends on functions chosen; typically small
Adaptation to orbit drift	Yes	No	Yes
Effect of high eccentricity	None	Requires excessive number of terms	None

drift significantly or undergo significant thrusting maneuvers, NGS requires an excessive number of interpolation points. An important example of such a system is a satellite that operates autonomously through orbit transfer.

Other (less attractive) methods of reducing the overhead of the EKF have been proposed. See Ref. 4 for details.

Conclusions

In general, autonav systems are time varying. Over the short term, the time dependence is periodic, but various parameters are subject to long-term drift or sudden changes due to thrusting. Even with the drifting parameters taken as constants, the resulting periodic systems cannot always be stabilized by a constant gain matrix. This was demonstrated for one example.

A new method of synthesizing time-varying gains (AGS) was offered. The designer guesses the structure of a special nonlinear coordinate transformation that makes the subsequent feedback design straightforward. Closed-form expressions for the gains and the closed-loop transition matrix are obtained as part of the process. In principle, AGS applies to any time-varying linear system, but some art is required; a suitable transformation must be found on a case-by-case basis. AGS is similar to pole placement for time-invariant systems in that the designer selects a number of parameters that serve as time constants for the closed-loop system.

RLS methods of time-varying feedback synthesis were compared with AGS. AGS solutions are preferred but are difficult to obtain. Numerical approximation of the RLS solution is less amenable to analysis but can always be done.

Future Work

This work raises a number of questions for further research: When can a time-varying system be stabilized by a constant gain matrix? What analysis and design techniques for time-invariant systems can be applied to time-varying systems through AGS? For what other space vehicle navigation problems is AGS appropriate? For what other time varying control problems is AGS appropriate?

Appendix A: Derivation of Dynamic Equations for the Example

We consider the system shown in Fig. 1. We derive the equations describing the complete fifth-order system and then specialize them to the case considered in this paper. The orbit is treated as Keplerian for design purposes, although a more accurate model may be selected for operation.

The kinematic equation is

$$\dot{\theta} = q + \dot{\nu} \tag{A1}$$

where θ is the pitch angle, q is the pitch angular velocity, and ν is the true anomaly. The rate of change of the true anomaly is given by

$$\dot{\nu}(n, \nu, e) = \frac{[1 + e\cos(\nu)]^2 n}{(1 - e^2)^{3/2}} \tag{A2}$$

where e is the eccentricity and n is the mean motion.

For the four orbit states, we choose n and a set of coordinates that are well behaved for any eccentricity: the mean argument of latitude Φ and the components A and B of the eccentricity vector. The nonlinear state equations are therefore

$$\dot{x}(n, \nu, e, q) = \begin{bmatrix} \dot{\theta} \\ \dot{n} \\ \dot{\Phi} \\ \dot{A} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} \dot{\nu}(n, \nu, e) + q \\ 0 \\ n \\ 0 \\ 0 \end{bmatrix} \tag{A3}$$

with the constraint equations

$$c(\nu, E, e, \omega, \Phi) = \begin{bmatrix} E - e\sin(E) - [\Phi - \omega] \\ \tan\left(\frac{E}{2}\right) - \left(\frac{1-e}{1+e}\right)^{1/2} \tan\frac{\nu}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{A4}$$

$$d(e, \omega, A, B) = \begin{bmatrix} A - e\cos(\omega) \\ B - e\sin(\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{A5}$$

The constraint equations are definitions of Φ, A, and B in terms of ν, e, the eccentric anomaly E, and the argument of perigee ω.

We define a nominal orbit, denoted by the subscript N, as

$$x_N = \begin{bmatrix} 0 \\ n(0) \\ \Phi(0) + n(0)t \\ A(0) \\ B(0) \end{bmatrix} \tag{A6}$$

Then the state equations linearized about the nominal orbit are

$$\begin{bmatrix} \dot{\theta} \\ \dot{\bar{n}} \\ \dot{\bar{\Phi}} \\ \dot{\bar{A}} \\ \dot{\bar{B}} \end{bmatrix} = \begin{bmatrix} \frac{\dot{\nu}_n}{n_N} \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \bar{n} + \begin{bmatrix} P_N \\ [0 \quad 0] \\ [0 \quad 0] \\ [0 \quad 0] \\ [0 \quad 0] \end{bmatrix} \begin{bmatrix} \bar{\nu} \\ \bar{e} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \bar{q} \tag{A7}$$

$$Q_N \begin{bmatrix} \bar{\nu} \\ \bar{E} \end{bmatrix} + R_N \begin{bmatrix} \bar{e} \\ \bar{\omega} \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \bar{\Phi} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{A8}$$

$$S_N \begin{bmatrix} \bar{e} \\ \bar{\omega} \end{bmatrix} + \begin{bmatrix} \bar{A} \\ \bar{B} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{A9}$$

where

$$P = \frac{\partial \dot{\nu}}{\partial(\nu, e)} \tag{A10}$$

$$Q = \frac{\partial c}{\partial(\nu, E)} \tag{A11}$$

$$R = \frac{\partial c}{\partial(e, \omega)} \tag{A12}$$

$$S = \frac{\partial d}{\partial(e, \omega)} \tag{A13}$$

and where the overbar indicates the difference between actual and nominal values:

$$\bar{x} = x - x_n \tag{A14}$$

Substitution from Eqs. (A8) and (A9) into Eq. (A7) yields

$$\dot{\theta} = \frac{\dot{\nu}_N}{n_N}\bar{n} + P_N \begin{bmatrix} [1 \quad 0] Q_N^{-1} R_N S_N^{-1} \\ [-1 \quad 0] S_N^{-1} \end{bmatrix} \begin{bmatrix} \bar{A} \\ \bar{B} \end{bmatrix}$$

$$+ P_N \begin{bmatrix} [1 \quad 0] Q_N^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \end{bmatrix} \bar{\Phi} + \bar{q} \tag{A15}$$

which finally evaluates to

$$\dot{\theta} = \frac{\dot{\nu}_N}{n_N}\bar{n} + U_N \begin{bmatrix} \bar{\Phi} \\ \bar{A} \\ \bar{B} \end{bmatrix} + \bar{q} \tag{A16}$$

where

$U = \dot{\nu} \times$

$$\begin{bmatrix} -\dfrac{2[1 + e\cos(\nu)]e\sin(\nu)}{(1 - e^2)^{3/2}} \\ -\dfrac{2[2 + e\cos(\nu)]A\sin^2(\nu)}{(1 - e^2)[1 + e\cos(\nu)]} - \dfrac{2[1 + e\cos(\nu)]\sin(\omega)\sin(\nu)}{(1 - e^2)^{3/2}} \\ + \dfrac{3A}{1 - e^2} + \dfrac{2\cos(\omega)\cos(\nu)}{1 + e\cos(\nu)} \\ -\dfrac{2[2 + e\sin(\nu)]B\sin^2(\nu)}{(1 - e^2)[1 + e\cos(\nu)]} + \dfrac{2[1 + e\cos(\nu)]\cos(\omega)\sin(\nu)}{(1 - e^2)^{3/2}} \\ + \dfrac{3B}{1 - e^2} + \dfrac{2\sin(\omega)\cos(\nu)}{1 + e\cos(\nu)} \end{bmatrix}^T \tag{A17}$$

For the example, $\Phi(0)$ and $B(0)$ are zero and $\bar{n}$, $\bar{\Phi}$, and $\bar{B}$ are artificially constrained to be zero. These simplifications yield the second-order state equations

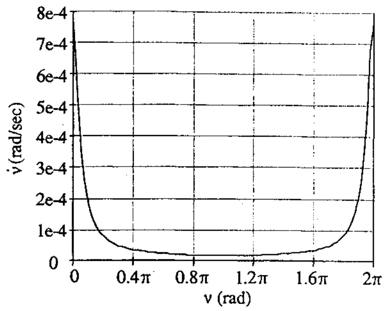

Fig. A1 Transformation from ν domain to time domain.

$$\begin{bmatrix} \dot{\theta} \\ \dot{\bar{e}} \end{bmatrix} = \begin{bmatrix} 0 & \dot{\nu}_N B(e_N) C(e_N, \nu_N) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \bar{e} \end{bmatrix} + \begin{bmatrix} \bar{q} \\ 0 \end{bmatrix} \tag{A18}$$

where

$$B = \frac{(2 + e)}{1 - e^2} \tag{A19}$$

$$C = \frac{2\cos(\nu) + e\cos(2\nu)}{2 + e} \tag{A20}$$

The subscripts N are no longer necessary; in the sequel they are omitted.

The estimator is Eq. (A18) with gyro data substituted for q and feedback of Earth sensor measurement of θ. It is modeled by

$$\begin{bmatrix} \dot{\tilde{\theta}} \\ \dot{\tilde{e}} \end{bmatrix} = \begin{bmatrix} -K_\theta(t) & \dot{\nu} BC \\ -K_e(t) & 0 \end{bmatrix} \begin{bmatrix} \tilde{\theta} \\ \tilde{e} \end{bmatrix} + \begin{bmatrix} v(t) \\ 0 \end{bmatrix} + \begin{bmatrix} K_\theta(t) \\ K_e(t) \end{bmatrix} w(t) \tag{A21}$$

where w and v are random errors and the tilde indicates estimate error:

$$\tilde{x} = \hat{x} - x \tag{A22}$$

It would serve the purposes of this paper (and afford a simpler derivation) to take ν and e as coordinates, assume $\omega = 0$ from the start, and set $\bar{\nu} = 0$ in the last step. The resulting equations of motion would be Eq. (A18) with

$$BC = \frac{3e}{1 - e^2} + \frac{2\cos(\nu)}{1 + e\cos(\nu)} \tag{A23}$$

We take the approach of simulating the full-order system using Φ, A, and B and allowing the simulated estimator to use the "truth" values of Φ and B. For this approach, the derivation given first is appropriate.

Although the simulation is run in the time domain, it is convenient to carry out the analysis with ν as the independent variable. Equation (21) is divided by $\dot{\nu}$ to yield the error equations

$$\begin{bmatrix} \tilde{\theta}' \\ \tilde{e}' \end{bmatrix} = \begin{bmatrix} -K_\theta(\nu) & BC \\ -K_e(\nu) & 0 \end{bmatrix} \begin{bmatrix} \tilde{\theta} \\ \tilde{e} \end{bmatrix} + \begin{bmatrix} v(\nu) \\ 0 \end{bmatrix} + \begin{bmatrix} K_\theta(\nu) \\ K_e(\nu) \end{bmatrix} w(\nu) \tag{A24}$$

where x' denotes the derivative of x with respect to ν and the gains are dimensionless.

Equation (24) is used to design $K_\theta(\nu)$ and $K_e(\nu)$. The filter gains are then calculated from

$$K_x(t) = \dot{\nu} K_x(\nu) \tag{A25}$$

This transformation is plotted in Fig. A1 for a LEO-GEO transfer orbit ($e = 0.7276$, $n = 8.2924 \times 10^{-5}$ rad/s). The transformation maps, for instance, a constant $K_\theta(\nu)$ into a time-varying $K_\theta(t)$, which is nearly zero for most of the orbit and spikes up near perigee.

Appendix B: Some Useful Tools

In Wei-Norman analysis, some skill is obviously required to pick a feasible set of m and F_i. Experience has shown that it is best to keep the rank of the F_i low, tending toward higher m.

Two classes of simple matrices are particularly useful for AGS analysis. They are 1) O_{ij}, a matrix whose (i, j)th element is one and whose other elements are zero and 2) $A_{ij} = O_{ij} - O_{ji}$, $i \neq j$. They are handy because their exponentials are simple and easily evaluated and products of the exponentials of any two of them, say O_{ij} and A_{kl}, tend to be identity unless some of i, j, k, and l are equal. This is seen in the identities

$$\exp(O_{ij} f) = \begin{cases} I + (e^f - 1) O_{ii}, & i = j \\ I + f O_{ij}, & i \neq j \end{cases} \tag{B1}$$

$$\begin{aligned} E(O_{i_1 j_1}, f, O_{i_2, j_2}) &= \exp(O_{i_1 j_1} f) O_{i_2 j_2} \exp(-O_{i_1 j_1} f) \\ &= O_{i_2 j_2} + \delta_{j_1 i_2}[f + \delta_{i_1 j_1}(e^f - 1 - f)] O_{i_1 j_2} \\ &\quad + \delta_{i_1 j_2}[-f + \delta_{i_1 j_1}(e^{-f} - 1 + f)] O_{i_2 j_1} \\ &\quad - \delta_{j_1 i_2}\delta_{i_1 j_2} f^2 O_{i_1 j_1} \\ &\quad + \delta_{i_1 j_1 i_2 j_2}(2 + f^2 - e^f - e^{-f}) O_{i_1 j_1} \end{aligned} \tag{B2}$$

$$\exp(A_{ij} f) = I + \sin(f) A_{ij} + [\cos(f) - 1](O_{ii} + O_{jj}) \tag{B3}$$

$$\begin{aligned} E(A_{i_1 j_1}, f, O_{i_2 j_2}) &= \exp(A_{i_1 j_1} f) O_{i_2 j_2} \exp(-A_{i_1 j_1} f) \\ &= O_{i_2 j_2} + (\delta_{i_1 j_2} + \delta_{j_1 i_2})[\cos(f) - 1] O_{i_2 j_2} \\ &\quad + \sin(f)(\delta_{j_1 i_2} O_{i_1 j_2} - \delta_{i_1 i_2} O_{j_1 j_2} \\ &\quad + \delta_{i_1 j_2} O_{i_2 j_2} - \delta_{j_1 j_2} O_{i_2 i_1}) \\ &\quad + \sin^2(f)[(\delta_{j_1 i_2}\delta_{i_1 j_2} + \delta_{i_1 i_2}\delta_{j_1 j_2}) O_{j_2 i_2} \\ &\quad - \delta_{i_1 i_2 j_2} O_{i_2 i_2} - \delta_{j_1 i_2 j_2} O_{i_1 i_1}] \\ &\quad + \sin(f)[\cos(f) - 1] \\ &\quad \times [(\delta_{j_1 i_2}\delta_{i_1 j_2} - \delta_{i_1 i_2}\delta_{j_1 j_2}) O_{j_2 j_2} \\ &\quad + \delta_{i_1 i_2 j_2} O_{j_1 j_2} - \delta_{j_1 i_2 j_2} O_{i_1 j_2}] \end{aligned} \tag{B4}$$

which follow from the definition of the matrix exponential. These formulas are rendered more readable in Tables B1 and B2, in which the δ's have been evaluated for particular cases.

The function

$$E(F_i, f, F_j) = \exp(F_i f) F_j \exp(-F_i f) \tag{B5}$$

is called the commutator product exponential because of the identity[2]

$$\exp(F_i) F_j \exp(-F_i) = F_j + [F_i, F_j] + \frac{[F_i, [F_i, F_j]]}{2!} + \frac{[F_i, [F_i, [F_i, F_j]]]}{3!} + \cdots \tag{B6}$$

where

$$[F_i, F_j] = F_i F_j - F_j F_i \tag{B7}$$

Table B1 Formulas for selected matrix exponentials

Condition	$\exp(O_{i_1 j_1} f)$	$E(O_{i_1 j_1}, f, O_{i_2 j_2})$
General	Eq. (B1)	Eq. (B2)
i_1, j_1, i_2, j_2 all different	$I + f O_{i_1 j_1}$	$O_{i_2 j_2}$
$i_1 = i_2, j_1 \neq i_1 \neq j_2 \neq j_1$	$I + f O_{i_1 j_1}$	$O_{i_2 j_2}$
$i_1 = j_2, j_1 \neq i_1 \neq i_2 \neq j_1$	$I + f O_{i_1 j_1}$	$O_{i_2 j_2} - f O_{i_2 j_1}$
$j_1 = i_2, i_1 \neq j_1 \neq j_2 \neq i_1$	$I + f O_{i_1 j_1}$	$O_{i_2 j_2} + f O_{i_1 j_2}$
$j_1 = j_2, i_1 \neq j_1 \neq i_2 \neq i_1$	$I + f O_{i_1 j_1}$	$O_{i_2 j_2}$
$i_1 = j_2, j_1 = i_2, i_1 \neq j_1$	$I + f O_{i_1 j_1}$	$O_{i_2 j_2} + f O_{i_1 j_2} - f O_{i_2 j_1} - f^2 O_{i_1 j_1}$
$i_1 = i_2, j_1 = j_2, i_1 \neq j_1$	$I + f O_{i_1 j_1}$	$O_{i_2 j_2}$
$i_1 = j_1 = i_2 \neq j_2$	$I + (e^f - 1) O_{i_1 i_1}$	$O_{i_2 j_2} + (e^f - 1) O_{i_1 j_2}$
$i_1 = j_1 = j_2 \neq i_2$	$I + (e^f - 1) O_{i_1 i_1}$	$O_{i_2 j_2} + (e^{-f} - 1) O_{i_2 j_1}$
$i_1 = i_2 = j_2 \neq j_1$	$I + f O_{i_1 j_1}$	$O_{i_2 j_2} - f O_{i_2 j_1}$
$j_1 = i_2 = j_2 \neq i_1$	$I + f O_{i_1 j_1}$	$O_{i_2 j_2} + f O_{i_1 j_2}$

Table B2 More formulas for selected matrix exponentials

Condition	$\exp(A_{i_1 j_1} f)$	$E(A_{i_1 j_1}, f, O_{i_2 j_2})$
General	Eq. (B3)	Eq. (B4)
i_1, j_1, i_2, j_2 all different	Eq. (B3)	$O_{i_2 j_2}$
$i_1 = i_2, j_1 \neq i_1 \neq j_2 \neq j_1$	Eq. (B3)	$O_{i_2 j_2} - \sin(f) O_{j_1 j_2}$
$i_1 = j_2, j_1 \neq i_1 \neq i_2 \neq j_1$	Eq. (B3)	$O_{i_2 j_2} + [\cos(f) - 1] O_{i_2 j_2} + \sin(f) O_{i_2 j_2}$
$j_1 = i_2, i_1 \neq j_1 \neq j_2 \neq i_1$	Eq. (B3)	$O_{i_2 j_2} + [\cos(f) - 1] O_{i_2 j_2} + \sin(f) O_{i_1 j_2}$
$j_1 = j_2, i_1 \neq j_1 \neq i_2 \neq i_1$	Eq. (B3)	$O_{i_2 j_2} - \sin(f) O_{i_2 i_1}$
$i_1 = j_2, j_1 = i_2, i_1 \neq j_1$	Eq. (B3)	$O_{i_2 j_2} + 2[\cos(f) - 1] O_{i_2 j_2} + \sin(f)(O_{i_1 j_2} + O_{i_2 j_2})$
$i_1 = i_2, j_1 = j_2, i_1 \neq j_1$	Eq. (B3)	$O_{i_2 j_2} + 2[\cos(f) - 1] O_{i_2 j_2} + \sin(f)(O_{i_1 j_2} + O_{i_2 j_2})$ $+ \sin^2(f) O_{j_2 i_2} + \sin(f)[\cos(f) - 1] O_{j_2 j_2})$
$i_1 = j_1 = i_2 \neq j_2$	Eq. (B3)	$O_{i_2 j_2} + [\cos(f) - 1] O_{i_2 j_2} + \sin(f)(O_{i_1 j_2} - O_{j_1 j_2})$
$i_1 = j_1 = j_2 \neq i_2$	Eq. (B3)	$O_{i_2 j_2} + [\cos(f) - 1] O_{i_2 j_2} + \sin(f)(O_{i_2 j_2} - O_{i_2 i_1})$
$i_1 = i_2 = j_2 \neq j_1$	Eq. (B3)	$O_{i_2 j_2} + [\cos(f) - 1] O_{i_2 j_2} + \sin(f)(O_{i_2 j_2} - O_{j_1 j_2})$ $+ \sin(f)[\cos(f) - 1] O_{j_1 j_2}$
$j_1 = i_2 = j_2 \neq i_1$	Eq. (B3)	$O_{i_2 j_2} + [\cos(f) - 1] O_{i_2 j_2} + \sin(f)(O_{i_1 j_2} - O_{i_2 i_1})$ $- \sin^2(f) O_{i_1 i_1} - \sin(f)[\cos(f) - 1] O_{i_1 j_2}$

is the commutator product bracket. For more complicated F_i, Eq. (B6) may be easier to apply than the definition of the matrix exponential.

The properties of the O_{ij}, for example, which make them useful as candidates for F_i, may be explained by group theory. See Ref. 5 for a related discussion.

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